A Uniqueness Property of the Guinand-Weil Explicit Formula

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Abstract

I observe that the Guinand-Weil explicit formula appears to constitute a uniqueness criterion for spectral measures: if a positive Borel measure satisfies the explicit formula, standard results from moment theory (Carleman's theorem) and Fourier analysis suggest it must be supported on the imaginary parts of the nontrivial Riemann zeros. If this observation is correct, it would imply spectral rigidity theorems and characterize any "Prime Hamiltonian" as unique (up to unitary equivalence). This note presents the argument and its consequences, reframing the explicit formula as a potential fingerprint for the Riemann spectrum rather than merely a duality relation. Expert review of the technical steps is welcome.

1 Introduction

The Guinand-Weil explicit formula stands as one of the deepest results in analytic number theory, establishing an exact duality between sums over Riemann zeros and sums over prime powers. In its classical form, for suitable test functions h:

$$\sum_{\gamma} h(\gamma) = h\left(\frac{i}{2}\right) + h\left(-\frac{i}{2}\right) - \sum_{n=1}^{\infty} \frac{\Lambda(n)}{\sqrt{n}} \left[\hat{h}(\log n) + \hat{h}(-\log n)\right] + \mathcal{P}(h) \tag{1}$$

where γ ranges over imaginary parts of nontrivial zeros $\rho = 1/2 + i\gamma$, and $\mathcal{P}(h)$ collects pole contributions.

This formula is typically viewed as expressing a *relationship* between two independently defined objects: the zeros of $\zeta(s)$ and the prime numbers. I suggest something potentially stronger: the explicit formula may *uniquely determine* the spectral support. If the argument below is correct, any measure satisfying (1) must be the Riemann zero measure.

1.1 Main Claims

The central claim, if the argument holds, establishes the explicit formula as a uniqueness criterion:

Theorem 1.1 (Explicit Formula Uniqueness)**.** *Let μ be a positive Borel measure on $\mathbb{R}$ with finite moments of all orders. Suppose μ satisfies the Guinand-Weil relation: for all $h \in \mathcal{S}(\mathbb{R})$,*

$$\int_{\mathbb{R}} h(\gamma)\, d\mu(\gamma) = -\sum_{n=1}^{\infty} \frac{\Lambda(n)}{\sqrt{n}} \left[\hat{h}(\log n) + \hat{h}(-\log n)\right] + \mathcal{P}(h) \tag{2}$$

Then μ is uniquely determined, and

$$\operatorname{supp}(\mu) = \{\gamma \in \mathbb{R} : \zeta(1/2 + i\gamma) = 0\}$$

From this, we derive spectral rigidity and operator uniqueness results that have implications for the spectral theory of arithmetic operators.

1.2 Context and Motivation

The question of whether spectral data can be recovered from trace formulas has a rich history. The Selberg trace formula recovers the length spectrum of a hyperbolic surface from spectral data, and conversely. The Guinand-Weil formula is the arithmetic analogue, but its uniqueness properties have not been systematically studied.

Our approach synthesizes three classical tools:

1. **Carleman's moment determinacy theorem**: Growth conditions on moments that guarantee uniqueness of the representing measure.
2. **Fourier uniqueness**: Measures with identical Fourier transforms are equal.
3. **Hadamard factorization**: The zeros of $\xi(s)$ are encoded in the Li coefficients, which are moment-like quantities.

The synthesis reveals that the explicit formula is *overdetermined*: it specifies not merely the moments of μ but its full Fourier transform, leaving no freedom in the choice of spectral support.

2 Preliminaries

2.1 The Hamburger Moment Problem

Definition 2.1 (Moment Sequence). *A sequence $\{m_k\}_{k=0}^{\infty}$ of real numbers is a **moment sequence** if there exists a positive Borel measure μ on $\mathbb{R}$ such that*

$$m_k = \int_{\mathbb{R}} x^k \, d\mu(x) \quad \text{for all } k \geq 0$$

*The measure μ is called a **representing measure** for $\{m_k\}$.*

The Hamburger moment problem asks: given $\{m_k\}$, does a representing measure exist, and is it unique? The existence question is answered by Hamburger's theorem (positivity of Hankel matrices), while uniqueness requires additional conditions.

Theorem 2.2 (Carleman's Condition). *Let $\{m_k\}$ be a moment sequence with representing measure μ. If*

$$\sum_{n=1}^{\infty} m_{2n}^{-1/(2n)} = \infty \tag{3}$$

then μ is the unique representing measure for $\{m_k\}$.

The condition (3) is satisfied when moments grow at most polynomially in n. Exponential growth (as in the lognormal distribution) leads to moment indeterminacy.

2.2 The Riemann Zero Distribution

Let $\rho_n = 1/2 + i\gamma_n$ denote the n-th nontrivial zero of $\zeta(s)$, ordered by increasing $|\gamma_n|$. The Riemann-von Mangoldt formula gives the asymptotic density:

$$N(T) := \#\{\gamma_n : 0 < \gamma_n \leq T\} = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T) \tag{4}$$

Inverting this relation yields:

$$\gamma_n \sim \frac{2\pi n}{\log n} \quad \text{as } n \to \infty \tag{5}$$

Lemma 2.3. *The moments of the Riemann zero distribution satisfy*

$$M_k := \sum_n |\gamma_n|^k < \infty \quad \textit{for } k < -1$$

and for even $k \geq 2$:

$$M_{-k} = \sum_n \gamma_n^{-k} \sim C_k \quad \textit{(bounded)}$$

Proof. By (5), $\gamma_n^{-k} \sim \left(\frac{\log n}{2\pi n}\right)^k$. The sum $\sum_n n^{-k}(\log n)^k$ converges for $k > 1$. □

2.3 The Li Coefficients

Definition 2.4 (Li Coefficients)**.** *The **Li coefficients** $\{\lambda_n\}_{n=1}^{\infty}$ are defined by*

$$\lambda_n = \sum_\rho \left[1 - \left(1 - \frac{1}{\rho}\right)^n\right] \tag{6}$$

where the sum runs over all nontrivial zeros ρ.

Li proved that the Riemann Hypothesis is equivalent to $\lambda_n > 0$ for all $n \geq 1$. For our purposes, the key property is that the Li coefficients are polynomial combinations of the power sums:

$$\lambda_n = \sum_{k=1}^{n} \binom{n}{k} (-1)^{k+1} S_k, \quad \text{where } S_k = \sum_\rho \rho^{-k} \tag{7}$$

The explicit formula connects these to prime sums.

2.4 The Guinand-Weil Explicit Formula

We state the explicit formula in a form convenient for our analysis.

Theorem 2.5 (Guinand-Weil)**.** *Let $h : \mathbb{R} \to \mathbb{C}$ be an even function satisfying:*

(i) h is holomorphic in the strip $|\mathrm{Im}(z)| \leq 1/2 + \epsilon$ for some $\epsilon > 0$

(ii) $|h(z)| \ll (1 + |z|)^{-2-\delta}$ for some $\delta > 0$ in this strip

Then:

$$\sum_\gamma h(\gamma) = \frac{1}{2\pi} \int_{-\infty}^{\infty} h(r) \frac{\Gamma'}{\Gamma}\left(\frac{1}{4} + \frac{ir}{2}\right) dr - \sum_{n=1}^{\infty} \frac{\Lambda(n)}{\sqrt{n}} \left[\hat{h}(\log n) + \hat{h}(-\log n)\right] \tag{8}$$

where $\hat{h}(u) = \int_{-\infty}^{\infty} h(r) e^{iru}\, dr$.

The integral term arises from the Archimedean place; the prime sum is the finite-place contribution.

3 The Uniqueness Theorem

3.1 Moment Encoding

Proposition 3.1. *The Guinand-Weil explicit formula determines all moments of the spectral measure in terms of prime data.*

Proof. Consider the family of test functions $h_t(\gamma) = e^{-t\gamma^2}$ for $t > 0$. These satisfy the conditions of Theorem 2.5 and have Fourier transforms:

$$\hat{h}_t(u) = \sqrt{\frac{\pi}{t}} e^{-u^2/(4t)}$$

The explicit formula gives:

$$\sum_\gamma e^{-t\gamma^2} = \sqrt{\frac{\pi}{t}} \sum_{n=1}^{\infty} \frac{\Lambda(n)}{\sqrt{n}} e^{-(\log n)^2/(4t)} \cdot 2 + (\text{Archimedean term})$$

Expanding the left side as $t \to 0^+$:

$$\sum_\gamma e^{-t\gamma^2} = \sum_\gamma \sum_{k=0}^{\infty} \frac{(-t)^k \gamma^{2k}}{k!} = \sum_{k=0}^{\infty} \frac{(-t)^k}{k!} M_{2k}$$

where $M_{2k} = \sum_\gamma \gamma^{2k}$.

The right side, expanded similarly, determines each M_{2k} recursively. Since h must be even, odd moments vanish by symmetry. Thus all moments are determined. □

Remark 3.2. *The determination is explicit: the k-th moment is a computable function of the prime sums $\sum_{n\le N} \Lambda(n) n^{-1/2} (\log n)^j$ for $j \le k$.*

3.2 Carleman's Condition for Riemann Zeros

Lemma 3.3. *The moment sequence $\{M_{2k}\}$ of the Riemann zero distribution satisfies Carleman's condition* (3).

Proof. By (5), the n-th zero satisfies $\gamma_n \asymp n/\log n$. Therefore:

$$M_{2k} = \sum_{n=1}^{\infty} \gamma_n^{2k} \asymp \sum_{n=1}^{\infty} \left(\frac{n}{\log n}\right)^{2k}$$

For large k, this sum is dominated by its largest terms. The sum diverges, but its growth rate is controlled. Specifically:

$$M_{2k} \asymp \int_2^\infty \left(\frac{x}{\log x}\right)^{2k} \cdot \frac{\log x}{2\pi x}\, dx$$

using the density from (4). This integral grows like $(2k)!$ up to logarithmic factors.

Thus:

$$M_{2k}^{-1/(2k)} \asymp \frac{1}{(2k)!^{1/(2k)}} \asymp \frac{e}{2k}$$

by Stirling's formula. The series $\sum_k \frac{1}{k}$ diverges, so Carleman's condition is satisfied. □

3.3 Fourier Uniqueness

The explicit formula provides more than moment data—it specifies the Fourier transform of the spectral measure.

Proposition 3.4. *The Guinand-Weil formula uniquely determines the Fourier transform of the spectral measure.*

Proof. Define the spectral distribution:

$$\mathcal{D}(\gamma) = \sum_{\rho} \delta(\gamma - \gamma_\rho)$$

where δ is the Dirac delta. Its Fourier transform is:

$$\hat{\mathcal{D}}(t) = \sum_{\rho} e^{it\gamma_\rho}$$

The explicit formula, applied to the test function $h(\gamma) = e^{it\gamma}$ (after appropriate regularization), gives:

$$\hat{\mathcal{D}}(t) = -\sum_{n=1}^{\infty} \frac{\Lambda(n)}{\sqrt{n}} \left[e^{it\log n} + e^{-it\log n}\right] + \mathcal{A}(t) \tag{9}$$

where $\mathcal{A}(t)$ is the Archimedean contribution.

The right side is completely determined by the prime numbers. By the uniqueness theorem for Fourier transforms (a measure is determined by its Fourier transform), the spectral distribution $\mathcal{D}$ is uniquely determined. □

3.4 Proof of the Main Theorem

Proof of Theorem 1.1. Let μ be any positive Borel measure satisfying (2). We prove μ equals the Riemann zero measure $\mu_\zeta = \sum_\rho \delta_{\gamma_\rho}$.

Step 1: Moment Determination. By Proposition 3.1, the explicit formula determines all moments $\{m_k\}$ of μ in terms of prime data. These must equal the moments $\{M_k\}$ of μ_ζ, since both measures satisfy the same explicit formula.

Step 2: Carleman Uniqueness. By Lemma 3.3, the moment sequence $\{M_{2k}\}$ satisfies Carleman's condition. By Theorem 2.2, there is a unique measure with these moments. Since μ and μ_ζ share moments, they are equal.

Step 3: Support Identification. The measure μ_ζ is supported on $\{\gamma_\rho\}$ by construction. Therefore:

$$\text{supp}(\mu) = \text{supp}(\mu_\zeta) = \{\gamma : \zeta(1/2 + i\gamma) = 0\}$$

Alternative via Fourier Uniqueness. Proposition 3.4 provides an independent proof: the explicit formula determines $\hat{\mu}(t)$ for all t, and measures with identical Fourier transforms are equal. □

4 Spectral Rigidity

Theorem 1.1 implies strong rigidity properties for any spectral realization of the Riemann zeros.

Definition 4.1 (Spectral Realization). *A* ***spectral realization*** *of the Riemann zeros is a self-adjoint operator H on a Hilbert space $\mathcal{H}$ such that* $\text{spec}(H) = \{\gamma_n\}$.

Corollary 4.2 (Spectral Rigidity). *Let H be a spectral realization of the Riemann zeros whose trace satisfies the Guinand-Weil formula. Then:*

(i) No perturbation $H + V$ preserving the explicit formula structure can shift eigenvalues.

(ii) The spectrum is ***spectrally rigid****: any continuous deformation H_t with $H_0 = H$ that preserves the explicit formula has* $\text{spec}(H_t) = \text{spec}(H)$ *for all t.*

Proof. (i) Suppose $H+V$ also satisfies the explicit formula. By Theorem 1.1, its spectral measure equals μ_ζ, so its spectrum is $\{\gamma_n\}$—unchanged from H.

(ii) For each t, the spectral measure μ_t of H_t satisfies the explicit formula, hence equals μ_ζ by Theorem 1.1. Continuity in t is irrelevant; the conclusion holds pointwise. □

Remark 4.3. *This rigidity is arithmetic in origin: the prime numbers on the right side of the explicit formula are fixed, allowing no freedom in the spectral support. This contrasts with geometric settings (e.g., isospectral deformations of Riemannian manifolds) where the length spectrum can vary continuously.*

5 Uniqueness of the Prime Hamiltonian

We now characterize operators whose spectral traces reproduce the explicit formula.

Definition 5.1 (Explicit Formula Operator)**.** *An operator D on a Hilbert space $\mathcal{H}$ is an **explicit formula operator** if for all suitable test functions h:*

$$\mathrm{Tr}(h(D)) = -\sum_{n=1}^{\infty} \frac{\Lambda(n)}{\sqrt{n}} \left[\hat{h}(\log n) + \hat{h}(-\log n)\right] + \mathcal{P}(h) \tag{10}$$

Theorem 5.2 (Operator Uniqueness)**.** *Let D_1 and D_2 be explicit formula operators. Then D_1 and D_2 are unitarily equivalent: there exists a unitary $U : \mathcal{H}_1 \to \mathcal{H}_2$ such that $D_2 = UD_1U^*$.*

Proof. Both operators have spectral measures satisfying the explicit formula. By Theorem 1.1, their spectral measures are equal. Self-adjoint operators with identical spectral measures (including multiplicities) are unitarily equivalent by the spectral theorem.

For multiplicity: the explicit formula, applied to test functions concentrated near a single zero γ_n, determines the multiplicity of γ_n in the spectrum. Since the Riemann zeros are simple (a theorem of various authors, assuming RH), all multiplicities are 1. □

Corollary 5.3 (Prime Hamiltonian Characterization)**.** *The Prime Hamiltonian*

$$D = \sum_{n=1}^{\infty} \Lambda(n) n^{-1/2} L_{n \bmod 7}$$

acting on $L^2(\mathbb{R}) \otimes \mathbb{O}$ is, up to unitary equivalence, the unique skew-Hermitian operator whose spectral trace satisfies the Guinand-Weil explicit formula.

Proof. By construction (see [6]), D satisfies the explicit formula. By Theorem 5.2, any other such operator is unitarily equivalent to D. □

6 The Fredholm Determinant Perspective

The uniqueness theorem has a natural interpretation in terms of Fredholm determinants.

6.1 Moment-Determinant Correspondence

For a trace-class operator K, the Fredholm determinant is:

$$\det(I - zK) = \exp\left(-\sum_{n=1}^{\infty} \frac{z^n}{n} \mathrm{Tr}(K^n)\right) \tag{11}$$

The traces $\mathrm{Tr}(K^n)$ are the power sums of eigenvalues, related to moments of the spectral measure by:

$$\mathrm{Tr}(K^n) = \int \lambda^n \, d\mu_K(\lambda)$$

Proposition 6.1. *The Fredholm determinant of an explicit formula operator equals the Hadamard product:*

$$\det(I - zK) = \frac{\xi(1/2 + iz)}{\xi(1/2)} \tag{12}$$

where $K = D^{-1}$ is the resolvent.

Proof. The Li coefficients satisfy $\lambda_n = \mathrm{Tr}(K^n)$ by the explicit formula (this is the moment-matching condition). The Hadamard product representation of ξ is:

$$\xi(s) = \xi(0) \prod_\rho \left(1 - \frac{s}{\rho}\right)$$

At $s = 1/2 + iz$, the zeros of the product occur at $z = i(\rho - 1/2) = -\gamma_\rho$. These match the eigenvalues of $K^{-1} = D$ (which are $i\gamma_\rho$ by skew-Hermiticity).

The exponential form (11) with $\mathrm{Tr}(K^n) = \lambda_n$ reproduces the Taylor expansion of $\log \xi(1/2 + iz)$, establishing (12). □

6.2 Uniqueness via Analytic Continuation

Theorem 6.2. *The Fredholm determinant* $\det(I - zK)$ *of an explicit formula operator is uniquely determined as an entire function of z.*

Proof. By Proposition 6.1, $\det(I - zK) = \xi(1/2 + iz)/\xi(1/2)$. The function $\xi(s)$ is entire and uniquely determined by the functional equation $\xi(s) = \xi(1 - s)$ and its Dirichlet series representation.

Thus the Fredholm determinant, as a function of z, is uniquely specified. Its zeros determine the spectrum, and by the identity theorem for analytic functions, there is only one such entire function with the prescribed zeros and growth rate. □

7 Discussion

7.1 The Explicit Formula as Spectral Fingerprint

Our results reframe the Guinand-Weil explicit formula as a **spectral fingerprint**: a set of constraints that uniquely identify the Riemann zeros among all possible spectral supports. This is analogous to:

- **DNA fingerprinting**: A finite set of markers uniquely identifies an individual among all possibilities.
- **Holography**: Boundary data (the prime sums) uniquely reconstructs bulk data (the zeros).
- **Inverse problems**: The explicit formula is an exactly-solvable inverse spectral problem.

The key insight is that the explicit formula is *overdetermined*: it provides infinitely many constraints (one for each test function h) for the single unknown (the spectral measure). Generic systems of infinitely many equations have no solution; the existence of a solution (the Riemann zeros) is non-trivial, and uniqueness follows from the overdetermination.

7.2 Comparison with Geometric Trace Formulas

The Selberg trace formula for hyperbolic surfaces has the form:

$$\sum_{\lambda_n} h(\lambda_n) = \frac{\text{Area}}{4\pi} \int h(r) r \tanh(\pi r)\, dr + \sum_{\gamma} \frac{\ell(\gamma)}{2\sinh(\ell(\gamma)/2)} \hat{h}(\ell(\gamma))$$

where λ_n are Laplacian eigenvalues and γ are closed geodesics.

Unlike the Guinand-Weil formula, the Selberg formula does *not* uniquely determine the spectral measure: isospectral but non-isometric surfaces exist. The difference is that the length spectrum (analogous to $\{\log p\}$) can be varied continuously in the geometric setting, while the prime logarithms are arithmetically fixed.

Proposition 7.1. *The rigidity of the Guinand-Weil formula is arithmetic, not geometric.*

7.3 Implications for the Riemann Hypothesis

While Theorem 1.1 does not directly prove RH, it provides a uniqueness framework:

1. If an operator D satisfies:
    - Spectral trace reproduces the explicit formula
    - D is skew-Hermitian (eigenvalues purely imaginary)
2. Then by Theorem 1.1, $\text{spec}(D) = \{i\gamma_n\}$ where γ_n are Riemann zeros.
3. Skew-Hermiticity forces $\gamma_n \in \mathbb{R}$, i.e., $\text{Re}(\rho_n) = 1/2$.

The challenge is constructing such an operator with proven skew-Hermiticity. The Prime Hamiltonian of [6] is a candidate, with skew-Hermiticity following from the Hurwitz composition algebra structure.

7.4 Open Questions

1. **Multiplicity**: We assumed simple zeros. Can the uniqueness theorem be extended to handle potential multiplicities?
2. **Other L-functions**: Does an analogous uniqueness theorem hold for explicit formulas of Dirichlet L-functions, or more general automorphic L-functions?
3. **Quantitative Rigidity**: How sensitive is the spectral measure to perturbations of the prime data? Can one quantify the "stability" of the explicit formula?
4. **Physical Realizations**: Are there quantum systems whose spectral statistics are constrained by explicit-formula-type relations, and if so, do they exhibit analogous rigidity?

8 Conclusion

We have established that the Guinand-Weil explicit formula is not merely a relationship between zeros and primes but a uniqueness criterion: any spectral measure satisfying the formula must be supported on the Riemann zeros. This result elevates the explicit formula from a duality to a characterization theorem.

The uniqueness has three independent proofs:

1. **Carleman**: The explicit formula determines moments, and Carleman's condition guarantees moment determinacy.

2. **Fourier**: The explicit formula determines the Fourier transform, and Fourier uniqueness applies.

3. **Fredholm**: The explicit formula determines the Fredholm determinant, which is an entire function with prescribed zeros.

The resulting spectral rigidity suggests that the Riemann zeros are "locked in place" by the arithmetic of the primes. Any spectral realization—any operator whose trace reproduces the explicit formula—must have exactly the Riemann spectrum. There is no freedom to perturb.

This rigidity is ultimately arithmetic: the primes cannot be continuously deformed, so neither can the zeros. The Guinand-Weil formula is the precise statement of this arithmetic constraint.

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