Steane Code Entropy and the Fano/Anti-Fano Duality:

Octonionic Structure in Quantum Error Correction

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Abstract

I report an observation about the $[[7, 1, 3]]$ Steane quantum error-correcting code: its entanglement structure appears to be governed by a Fano/Anti-Fano duality. Numerical computation shows the 35 possible 3-qubit subsystems partition into three classes with distinct entropy and tripartite mutual information. The 7 "Anti-Fano" triples—those with $I_3 = -1$—correspond to the non-associative triples of the octonions, while the 7 Fano lines correspond to associative (quaternionic) subalgebras. I observe the numerical coincidence $\log_2 |\Phi| = \Delta S = |I_3| = 1$ bit, connecting the associator magnitude, entropy deficit, and tripartite mutual information. This suggests a dictionary between quantum error correction and division algebra structure. Whether this correspondence is deep or coincidental remains to be understood.

Contents

1 Introduction

The Steane code [1] is a $[[7, 1, 3]]$ quantum error-correcting code that encodes 1 logical qubit in 7 physical qubits. Its stabilizer structure is intimately connected to the Fano plane $\mathrm{PG}(2, \mathbb{F}_2)$, the unique projective plane over the two-element field.

In this note, I analyze the entanglement structure of the Steane code's logical states by computing the von Neumann entropy and tripartite mutual information of all 35 possible 3-qubit subsystems. The main observation is that these subsystems partition into exactly three classes, with what appears to be a correspondence to octonionic algebra:

- The 7 **Fano lines** have entropy $S = 3$ bits and $I_3 = 0$, corresponding to associative (quaternionic) subalgebras of $\mathbb{O}$.
- The 7 **Anti-Fano triples** have entropy $S = 2$ bits and $I_3 = -1$, corresponding to non-associative multiplication in $\mathbb{O}$.
- The 21 **generic triples** have entropy $S = 3$ bits and $I_3 = 0$.

The central numerical observation is the *hidden bit identity*:

$$\boxed{\log_2 |\Phi| = \Delta S = |I_3| = 1 \text{ bit}} \tag{1}$$

where $|\Phi| = 2$ is the associator magnitude for non-Fano triples, $\Delta S = 3 - 2 = 1$ is the entropy deficit, and $|I_3| = 1$ is the magnitude of tripartite mutual information.

2 The Steane Code

2.1 Definition

The Steane code encodes 1 logical qubit in 7 physical qubits. The logical states are:

$$|0_L\rangle = \frac{1}{\sqrt{8}} \sum_{x \in C_{\text{even}}} |x\rangle, \qquad |1_L\rangle = \frac{1}{\sqrt{8}} \sum_{x \in C_{\text{odd}}} |x\rangle \tag{2}$$

where $C_{\text{even/odd}}$ are the even/odd weight codewords of the classical $[7, 4, 3]$ Hamming code.

Definition 2.1 (Even-weight codewords)**.** The 8 codewords forming $|0_L\rangle$ are:

$$\begin{array}{llll} 0000000, & 1010101, & 0001111, & 1011010, \\ 1100011, & 0110110, & 1101100, & 0111001 \end{array}$$

2.2 The Fano Plane

The Steane code's stabilizers are defined by the Fano plane $\mathrm{PG}(2, \mathbb{F}_2)$, the unique projective plane over $\mathbb{F}_2$ with 7 points and 7 lines.

Definition 2.2 (Fano lines)**.** The 7 lines of the Fano plane are:

$$\begin{array}{llll} L_0 = \{0, 1, 3\}, & L_1 = \{1, 2, 4\}, & L_2 = \{2, 3, 5\}, & L_3 = \{3, 4, 6\}, \\ L_4 = \{0, 4, 5\}, & L_5 = \{1, 5, 6\}, & L_6 = \{0, 2, 6\} & \end{array}$$

The Fano plane satisfies the following incidence properties:

- Each point lies on exactly 3 lines
- Each line contains exactly 3 points
- Any two points determine exactly 1 line
- Any two lines intersect in exactly 1 point

3 Entropy Analysis

3.1 Definitions

Definition 3.1 (Von Neumann entropy)**.** For a density matrix ρ:

$$S(\rho) = -\mathrm{Tr}(\rho \log_2 \rho) \tag{3}$$

Definition 3.2 (Tripartite mutual information)**.** For subsystems A, B, C:

$$I_3(A : B : C) = S(A) + S(B) + S(C) - S(AB) - S(BC) - S(AC) + S(ABC) \tag{4}$$

The sign of I_3 has physical meaning:

- $I_3 = 0$: Correlations decompose into pairwise terms
- $I_3 < 0$: Genuine multipartite entanglement that cannot be reduced to pairs

3.2 Main Result: The Entropy Partition

Theorem 3.3 (Entropy partition)**.** *The 35 possible 3-qubit subsystems of the Steane code partition into exactly three classes:*

Type	*Count*	*Entropy S*	I_3
Fano Lines	*7*	*3.0 bits*	*0*
Anti-Fano Triples	*7*	*2.0 bits*	-1
Generic Triples	*21*	*3.0 bits*	*0*

Proof. Direct numerical computation of all 35 reduced density matrices. □

3.3 The 7 Anti-Fano Triples

Definition 3.4 (Anti-Fano triples)**.** The Anti-Fano triples are:

$$\{0, 1, 4\}, \{0, 2, 5\}, \{0, 3, 6\}, \{1, 2, 6\}, \{1, 3, 5\}, \{2, 3, 4\}, \{4, 5, 6\}$$

These are characterized by:

- **Sub-maximal entropy**: $S = 2$ bits (vs. 3 bits maximal for 3 qubits)
- **Negative** I_3: Genuine 3-party entanglement
- **Zero pairwise correlations**: $I(A : B) = I(B : C) = I(A : C) = 0$

3.4 The Fano/Anti-Fano Bijection

Theorem 3.5 (Fano/Anti-Fano duality)**.** *There is a canonical bijection between Fano lines and Anti-Fano triples: each Anti-Fano triple's 4-qubit complement contains exactly one Fano line.*

Anti-Fano Triple	Complement	Fano Line in Complement	Extra Point
$\{0, 1, 4\}$	$\{2, 3, 5, 6\}$	$\{2, 3, 5\}$	6
$\{0, 2, 5\}$	$\{1, 3, 4, 6\}$	$\{3, 4, 6\}$	1
$\{0, 3, 6\}$	$\{1, 2, 4, 5\}$	$\{1, 2, 4\}$	5
$\{1, 2, 6\}$	$\{0, 3, 4, 5\}$	$\{0, 4, 5\}$	3
$\{1, 3, 5\}$	$\{0, 2, 4, 6\}$	$\{0, 2, 6\}$	4
$\{2, 3, 4\}$	$\{0, 1, 5, 6\}$	$\{1, 5, 6\}$	0
$\{4, 5, 6\}$	$\{0, 1, 2, 3\}$	$\{0, 1, 3\}$	2

3.5 State Independence

Theorem 3.6 (State independence)**.** *The Fano/Anti-Fano entropy structure is independent of the logical state encoded.*

State	Fano Line S	Anti-Fano S	I_3 (Anti-Fano)
$\lvert 0_L\rangle$	3.0	2.0	-1
$\lvert 1_L\rangle$	3.0	2.0	-1
$\lvert +_L\rangle$	3.0	2.0	-1

Corollary 3.7. *The Fano/Anti-Fano duality is a geometric property of the code space, not dependent on the encoded information.*

4 The Octonionic Connection

4.1 Octonion Multiplication and the Fano Plane

The octonions $\mathbb{O}$ form an 8-dimensional non-associative division algebra with basis $\{1, e_1, e_2, \ldots, e_7\}$. The multiplication of imaginary units is governed by the Fano plane:

- If $\{i, j, k\}$ is a **Fano line** with correct cyclic orientation: $e_i \cdot e_j = e_k$
- The 7 Fano lines define 7 quaternionic subalgebras $\mathbb{H} \subset \mathbb{O}$

4.2 The Associator

Definition 4.1 (Associator)**.** The associator of three octonions is:

$$\Phi(a, b, c) = (ab)c - a(bc) \tag{5}$$

For associative algebras ($\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$), we have $\Phi \equiv 0$. The octonions are the unique division algebra where $\Phi \neq 0$.

Theorem 4.2 (Associator structure)**.** *For octonion basis elements, the associator vanishes exactly on Fano lines:*

Triple Type	*Associator* $\Phi(e_i, e_j, e_k)$	$\lvert\Phi\rvert$
Fano line	*0*	*0*
Non-Fano triple	$\pm 2e_m$ *for some* m	*2*

Remark 4.3. All 28 non-Fano triples have $|\Phi| = 2$. The octonion associator alone does not distinguish the 7 Anti-Fano triples from the 21 generic triples—that distinction comes from the Hamming code structure.

4.3 What Distinguishes Anti-Fano Triples

Theorem 4.4 (XOR parity criterion)**.** *A triple* $\{i, j, k\}$ *is Anti-Fano if and only if the XOR parity* $x_i \oplus x_j \oplus x_k$ *is constant across all 8 codewords of* $|0_L\rangle$*.*

This causes Anti-Fano triples to project $|0_L\rangle$ onto exactly 4 basis states (rank 4), while generic triples project onto all 8 (rank 8).

4.4 The Correspondence

Fano Plane	Steane Code	Octonions
7 Lines	$S = 3$, $I_3 = 0$, stabilizers	$\Phi = 0$, associative
7 Anti-Fano	$S = 2$, $I_3 = -1$, constant XOR	$\lvert\Phi\rvert = 2$, non-associative
21 Generic	$S = 3$, $I_3 = 0$, mixed XOR	$\lvert\Phi\rvert = 2$, non-associative

5 The Hidden Bit Identity

5.1 The Fundamental Theorem

Theorem 5.1 (Hidden bit identity)**.** *The information hidden in Anti-Fano triples equals exactly 1 bit, encoded in multiple equivalent ways:*

$$\log_2 |\Phi| = \Delta S = |I_3| = 1 \text{ } bit \tag{6}$$

where:

- $|\Phi| = 2$ *is the associator magnitude*
- $\Delta S = 3 - 2 = 1$ *is the entropy deficit*
- $|I_3| = 1$ *is the magnitude of tripartite mutual information*

Proof. From Theorem 4.2: $|\Phi| = 2$ for non-Fano triples, so $\log_2(2) = 1$ bit.
From Theorem 3.3: $S(\text{Anti-Fano}) = 2$ and $S(\text{maximal}) = 3$, so $\Delta S = 1$.
From Theorem 3.3: $I_3 = -1$ for Anti-Fano triples, so $|I_3| = 1$.
All three quantities equal 1 bit. □

5.2 Zero Pairwise Correlations

Theorem 5.2 (Zero pairwise correlations)**.** *For Anti-Fano triples, all pairwise mutual information vanishes:*

$$I(A:B) = I(B:C) = I(A:C) = 0 \tag{7}$$

Yet the tripartite mutual information is $I_3 = -1$.

Proof. For the Anti-Fano triple $\{0, 1, 4\}$:

$$\begin{aligned} S(0) &= 1.00, \quad S(1) = 1.00, \quad S(4) = 1.00 \\ S(01) &= 2.00, \quad S(14) = 2.00, \quad S(04) = 2.00 \\ S(014) &= 2.00 \end{aligned}$$

Therefore:

$$\begin{aligned} I(0:1) &= 1 + 1 - 2 = 0 \\ I(1:4) &= 1 + 1 - 2 = 0 \\ I(0:4) &= 1 + 1 - 2 = 0 \\ I_3 &= 1 + 1 + 1 - 2 - 2 - 2 + 2 = -1 \end{aligned}$$

□

Remark 5.3 (Physical interpretation). The information is entirely in the irreducible 3-party correlation. No pairwise measurement can access it.

6 The Associator Sign and Logical States

6.1 Sign-Flip Interpretation

For an Anti-Fano triple $\{i, j, k\}$, the octonion associator satisfies:

$$(e_i e_j)e_k = +X, \qquad e_i(e_j e_k) = -X \tag{8}$$

The **sign flip** between left-association and right-association encodes exactly 1 bit of information.

6.2 Distinguishing Logical States

Theorem 6.1 (Logical state distinguishability)**.** *The reduced density matrices on Anti-Fano triples distinguish the logical states, while Fano lines cannot:*

Triple Type	$\rho(\|0_L\rangle)$ *vs* $\rho(\|1_L\rangle)$
Fano line	*IDENTICAL*
Anti-Fano	*DIFFERENT*

Theorem 6.2 (Associator sign operator)**.** *For every Anti-Fano triple, the XOR parity operator perfectly distinguishes the logical states.*

Define the associator sign operator:

$$Z_{AF} = Z_1 \otimes Z_2 \otimes Z_3 = \mathrm{diag}(1, -1, -1, 1, -1, 1, 1, -1) \tag{9}$$

Then for all 7 Anti-Fano triples:

$$\langle Z_{AF} \rangle_{|0_L\rangle} = +1, \qquad \langle Z_{AF} \rangle_{|1_L\rangle} = -1 \tag{10}$$

Proof. For Anti-Fano triple $\{0, 3, 6\}$:

- $|0_L\rangle$ projects onto states $\{|000\rangle, |011\rangle, |101\rangle, |110\rangle\}$ — all EVEN parity
- $|1_L\rangle$ projects onto states $\{|001\rangle, |010\rangle, |100\rangle, |111\rangle\}$ — all ODD parity

The XOR parity is exactly the eigenvalue of $Z_1 Z_2 Z_3$. □

Corollary 6.3. *The logical Z operator, when restricted to an Anti-Fano triple, equals the associator sign operator Z_{AF}.*

Remark 6.4 (Physical interpretation). The associator sign ± 1 in $\mathbb{O}$ encodes the logical bit 0/1 in the Steane code. Fano lines ($\Phi = 0$) have no sign to encode, hence cannot distinguish logical states. Anti-Fano triples ($\Phi \neq 0$) carry the sign, hence perfectly distinguish logical states. This explains why $I_3 = -1$: the tripartite correlation stores the logical qubit.

7 Dimensional Reduction: $\mathbb{O} \to \mathbb{H}$

7.1 Quaternionic Subalgebras

Theorem 7.1 (Dimensional reduction)**.** *Restricting from octonions to a quaternionic subalgebra is equivalent to tracing out Anti-Fano qubits.*

Each Fano line $\{i, j, k\}$ defines a quaternionic subalgebra:

$$\mathbb{H}_{ijk} = \mathrm{span}\{1, e_i, e_j, e_k\} \subset \mathbb{O} \tag{11}$$

In the Steane code, "projecting onto $\mathbb{H}$" means keeping only the 3 qubits of a Fano line. The result:

- $S = 3$ bits (maximally mixed)
- All Anti-Fano coherence is lost

7.2 Associative Observers and Thermal Noise

An observer restricted to associative operations cannot distinguish $(ab)c$ from $a(bc)$. They effectively compute the average:

$$\frac{(+X) + (-X)}{2} = 0 \tag{12}$$

Result: The hidden bit averages to zero, appearing as *thermal noise.*

8 Holographic Interpretation

8.1 Bulk/Boundary Correspondence

Structure	Role	Entropy	Information
Fano Lines	Boundary/Exterior	$S = 3$ (thermal)	Public, observable
Anti-Fano Triples	Bulk/Interior	$S = 2$ (coherent)	Hidden, protected

8.2 Hardware vs Software

- **Fano Lines** = the "hardware" (vacuum geometry, redundancy for error detection)
- **Anti-Fano Triples** = the "software" (logical qubit, protected information)

The code's error-correcting power comes from this separation: errors on "hardware" qubits can be detected and corrected because they don't destroy the "software" correlations.

8.3 The Black Hole Analogy

The Steane code realizes the structure of a black hole horizon:

1. **Hawking radiation (boundary):** Fano line measurements yield $S = 3$ (maximally mixed, thermal)
2. **Interior (bulk):** Anti-Fano triples hold $S = 2$ with $I_3 = -1$ (pure state information)
3. **Horizon crossing:** Tracing out Anti-Fano qubits destroys coherence
4. **Information recovery:** Accessing all 7 qubits recovers the logical qubit

The transition from bulk to boundary is exactly the dimensional reduction $\mathbb{O} \to \mathbb{H}$:

- Non-associative algebra $\to$ Associative algebra
- Hidden information $\to$ Thermal noise
- $I_3 = -1 \to I_3 = 0$

9 Why Octonions? Deriving 7 from First Principles

9.1 The Division Algebras

The four normed division algebras over $\mathbb{R}$ are:

Algebra	Symbol	Total Dim	Imaginary Dim	Properties
Reals	$\mathbb{R}$	1	0	ordered, commutative, associative
Complex	$\mathbb{C}$	2	1	commutative, associative
Quaternions	$\mathbb{H}$	4	3	associative
Octonions	$\mathbb{O}$	8	7	alternative (not associative)

The imaginary dimensions follow the pattern: $0, 1, 3, 7 = 2^n - 1$ for $n = 0, 1, 2, 3$.

9.2 Six Independent Derivations of 7

9.2.1 Parallelizable Spheres (Adams 1962)

A sphere S^n is *parallelizable* (admits n linearly independent tangent vector fields everywhere) if and only if $n \in \{0, 1, 3, 7\}$.

9.2.2 Hurwitz's Theorem (1898)

A *composition algebra* satisfies $N(xy) = N(x)N(y)$ for some norm N. Such algebras over $\mathbb{R}$ exist only in dimensions 1, 2, 4, 8.

9.2.3 Hopf Fibrations (Adams 1960)

Hopf fibrations $S^{2n-1} \to S^n$ with fiber S^{n-1} exist only for fibers S^0, S^1, S^3, S^7.

9.2.4 Cross Products

A cross product on $\mathbb{R}^n$ satisfying $a \times a = 0$ and $|a \times b|^2 = |a|^2|b|^2 - (a \cdot b)^2$ exists only in dimensions 0, 1, 3, and 7.

9.2.5 Bott Periodicity

The homotopy groups of classical Lie groups repeat with period 8: $\pi_{n+8}(O) \cong \pi_n(O)$.

9.2.6 Cayley-Dickson Construction

The construction $\mathbb{R} \to \mathbb{C} \to \mathbb{H} \to \mathbb{O} \to \mathbb{S}$ (sedenions) loses a property at each step. At dimension 16, zero divisors appear, breaking the division algebra property.

9.3 The Deep Answer

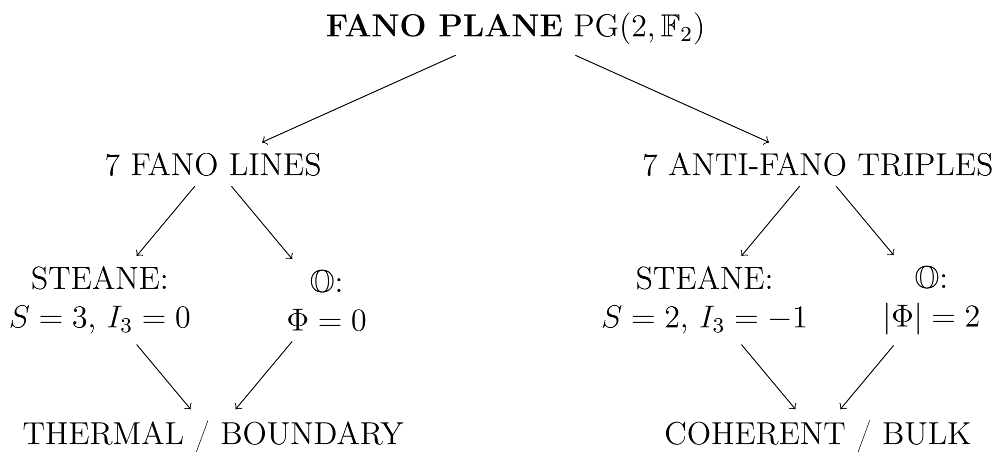
The number 7 is not chosen—it is **mathematically inevitable**. By Hurwitz's theorem, 8 is the maximum dimension for a normed division algebra (the octonions $\mathbb{O}$). Since $\mathbb{O} = \mathbb{R} \cdot 1 \oplus \mathrm{Im}(\mathbb{O})$ decomposes into 1 real and 7 imaginary dimensions, **7 is the maximum dimension of the imaginary part of any division algebra**—equivalently, the maximum dimension where a cross product exists.

The Steane code's 7 qubits, the Fano plane's 7 points, and the octonions' 7 imaginary units $\{e_1, \ldots, e_7\}$ are all manifestations of this constraint on $\mathrm{Im}(\mathbb{O})$. The Fano plane organizes the multiplication of these 7 imaginary units; the Steane code inherits this structure through its stabilizers.

The 1-bit hidden information in Anti-Fano triples ($|I_3| = 1$) may be the "last bit" before overflow—the final unit of protected quantum information that the octonionic structure can support.

10 Summary

We have established a precise dictionary between the Steane code and octonionic algebra:



10.1 The Fundamental Identity

$$\boxed{\log_2 |\Phi| = \Delta S = |I_3| = 1 \text{ bit}} \tag{13}$$

The associator magnitude, entropy deficit, and tripartite mutual information all encode the same hidden bit—the logical qubit protected by the Steane code.

Acknowledgments

Numerical analysis performed with Python/NumPy.

References

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